Available online at www.sciencedirect.com
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J. Math. Anal. Appl. 305 (2005) 107–120
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Impulsive delay differential inequality and stability of neural networks ☆

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Received 3 February 2004
Available online 23 November 2004
Submitted by I. Lasiecka

Abstract

In this article, a generalized model of neural networks involving time-varying delays and impulses is considered. By establishing the delay differential inequality with impulsive initial conditions and using the properties of M-cone and eigenspace of the spectral radius of nonnegative matrices, some new sufficient conditions for global exponential stability of impulsive delay model are obtained. The results extend and improve the earlier publications. An example is given to illustrate the theory.

Keywords: Impulse; Delay; Differential inequality; Neural networks; Stability

1. Introduction

Stability for neural networks has attracted considerable attention due to its important role in designs and applications of the networks. In particular, delay effect on the stability and other dynamical behaviors of neural networks has been extensively studied in the literature, e.g., Refs. [1–12].

☆ The work is supported by National Natural Science Foundation of China under Grant 10371083.
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.10.040

However, besides delay effect, impulsive effect likewise exists in a wide variety of evolutionary processes in which states are changed abruptly at certain moments of time, involving such fields as medicine and biology, economics, mechanics, electronics and telecommunications, etc. Many interesting results on impulsive effect have been gained, e.g., Refs. [14–20]. As artificial electronic systems, neural networks such as Hopfield neural networks, bidirectional neural networks and recurrent neural networks often are subject to impulsive perturbations which can affect dynamical behaviors of the systems just as time delays. Therefore, it is necessary to consider both impulsive effect and delay effect on the stability of neural networks. And yet few results have been developed in the direction for neural networks (Guan [18]).

In this paper, on the basis of the structure of Hopfield neural networks and recurrent neural networks, we consider a class of general neural networks described by impulsive delay differential equations:

$$\begin{cases} y_i'(t) = -c_i y_i(t) + \sum_{j=1}^n a_{ij} F_j(y_j(t)) + \sum_{j=1}^n b_{ij} G_j(y_j(t-\tau_{ij}(t))) + I_i, \\ \quad t \neq t_k, \\ y_i(t) = H_{ik}(y_1(t^-), \dots, y_n(t^-)) \\ \qquad\qquad + W_{ik}(y_1((t-\tau_{i1}(t))^-), \dots, y_n((t-\tau_{in}(t))^-)) + J_{ik}, \\ \quad t = t_k, \end{cases} \tag{1}$$

where $c_i > 0$, $0 \leqslant \tau_{ij}(t) \leqslant \tau$, the fixed moments of time t_k satisfy $t_1 < t_2 < \dots$, $\lim_{k\to\infty} t_k = \infty$, $i, j = 1, \dots, n$ and $k = 1, 2, \dots$. The first part (called continuous part) of model (1) describes the continuous evolution processes of the neural networks. n corresponds to the number of units in the neural networks, y_i corresponds to the state variable, $F_j(x_j), G_j(x_j)$ are the activation functions of the neurons, c_i represents the neuron charging time constant, a_{ij}, b_{ij} stand for the weights of the neuron interconnections, I_i is the external bias, and $\tau_{ij}(t)$ corresponds to the transmission delay. The second part (called discrete part) of model (1) describes that the evolution processes experience abrupt change of states at the moments of time t_k (called impulsive moments). $H_{ik}(y_1(t^-), \dots, y_n(t^-))$ represents impulsive perturbations of the ith unit at time t_k and $y_j(t^-)$ denotes the left limit of $y_j(t)$ for $j = 1, \dots, n$, $W_{ik}(y_1((t-\tau_{i1}(t))^-), \dots, y_n((t-\tau_{in}(t))^-))$ denotes impulsive perturbations of the ith unit at time t_k which caused by the transmission delays, J_{ik} represents external impulsive input at time t_k.

If $H_{ik}(y_1, \dots, y_n) = y_i$, $W_{ik}(y_1, \dots, y_n) = 0$, $J_{ik} = 0$ for $i = 1, \dots, n$, $k = 1, 2, \dots$, then the system (1) becomes the nonimpulsive neural networks, which contains many popular models such as Hopfield neural networks, bidirectional neural networks, cellular neural networks and recurrent neural networks, etc. (see, e.g., [3–12,31]).

The main difficulty for stability analysis of model (1) comes from both impulsive effect and delay effect on the system since the corresponding theory for impulsive delay differential equations have not yet been fully developed. Although some authors (e.g., Anokhin [13], Bainov and Simenov [14], Lakshmikantham et al. [15], Liu [20], etc.) have established some criteria on stability for impulsive differential equations using Lyapunov functions, the skillful construction of Lyapunov functions to obtain the stability of nonimpulsive delay neural networks system (see Zhou [5], Sun [8], Cao [11], Guo [12], etc.) may be difficult and even ineffective for the impulsive delay neural networks. Another important tool for investigating dynamical behavior of differential equation is the differential inequality. However, existing results in [25–30] are also ineffective for the impulsive delay neural networks since they require at least that the solutions of differential inequalities are continuous. Therefore, techniques and methods for stability of impulsive delay differential systems should be developed and explored. This paper presents one such method by establishing an impulsive delay differential inequality and employing M-cone and eigenspace of the spectral radius of nonnegative matrices. Based on the obtained method, we shall give sufficient conditions for the globally exponential stability of the equilibrium point of impulsive delay neural networks (1). The results can extend and improve the recent works. An example is given to demonstrate the effectiveness of the results.

2. Preliminaries

To begin with, we introduce some notation and recall some basic definitions.

Let R^n be the space of n-dimensional real column vectors and $R^{m\times n}$ denote the set of $n\times m$ real matrices. Usually E denotes an $n\times n$ unit matrix and $e_n=(1,\dots,1)^T\in R^n$. For $A,B\in R^{m\times n}$ or $A,B\in R^n$, $A\geqslant B(A\leqslant B, A>B, A<B)$ means that each pair of corresponding elements of A and B satisfies the inequality "$\geqslant(\leqslant,>,<)$." Especially, A is called a nonnegative matrix if $A\geqslant 0$, and z is called a positive vector if $z>0$.

For $x=(x_1,\dots,x_n)^T\in R^n$, we denote

$$\operatorname{Sgn}(x)=\operatorname{diag}\bigl\{\operatorname{sgn}(x_1),\dots,\operatorname{sgn}(x_n)\bigr\},\qquad \operatorname{sgn}(x_i)=\begin{cases}-1, & \text{if } x_i<0,\\ 0, & \text{if } x_i=0,\\ 1, & \text{if } x_i>0.\end{cases}$$

$C[X,Y]$ denotes the space of continuous mappings from the topological space X to the topological space Y. Especially, let $C\triangleq C[[-\tau,0],R^n]$, where $\tau>0$.

$PC[I,R^n]\triangleq\{\psi: I\to R^n \mid \psi(t^+)=\psi(t)$ for $t\in I$, $\psi(t^-)$ exists for $t\in(t_0,\infty)$, $\psi(t^-)=\psi(t)$ for all but points $t_k\in(t_0,\infty)\}$, where $I\subset R$ is an interval, $\psi(t^+)$ and $\psi(t^-)$ denote the left-hand and right-hand limits of scalar function $\psi(t)$, respectively. Especially, let $PC=PC([-\tau,0],R^n)$.

For $x\in R^n$, $A\in R^{n\times n}$, $\varphi\in C$ or $\varphi\in PC$, we define

$$[x]^+=\bigl(|x_1|,\dots,|x_n|\bigr)^T,\qquad [A]^+=\bigl(|a_{ij}|\bigr)_{n\times n},$$
$$\bigl[\varphi(t)\bigr]_\tau=\bigl(\bigl[\varphi_1(t)\bigr]_\tau,\dots,\bigl[\varphi_n(t)\bigr]_\tau\bigr)^T,\qquad \bigl[\varphi(t)\bigr]_\tau^+=\bigl[\bigl[\varphi(t)\bigr]^+\bigr]_\tau,$$

where $[\varphi_i(t)]_\tau=\sup_{-\tau\leqslant s\leqslant 0}\{\varphi_i(t+s)\}$. And we introduce the corresponding norm for them as follows:

$$\|x\|=\max_{1\leqslant i\leqslant n}\bigl\{|x_i|\bigr\},\qquad \|A\|=\max_{1\leqslant i\leqslant n}\sum_{j=1}^n|a_{ij}|,\qquad \|\varphi\|=\max_{1\leqslant i\leqslant n}\bigl\{\bigl[\varphi_i(t)\bigr]_\tau^+\bigr\}.$$

For convenience, we shall rewrite system (1) in the vector form:

$$\begin{cases} y'(t)=-Cy(t)+AF(y(t))+BG(y(t-\tau(t)))+I, & t\neq t_k,\\ y(t)=H_k(y(t^-))+W_k(y((t-\tau(t))^-))+J_k, & t=t_k,\end{cases}\tag{2}$$

where $y(t)=(y_1(t),\dots,y_n(t))^T$, $C=\operatorname{diag}\{c_1,\dots,c_n\}$, $A=(a_{ij})_{n\times n}$, $B=(b_{ij})_{n\times n}$, $I=(I_1,\dots,I_n)^T$, $J_k=(J_{1k},\dots,J_{nk})^T$, $F(y)=(F_1(y_1),\dots,F_n(y_n))^T$, $G(y)=(G_1(y_1),\dots,G_n(y_n))^T$, $H_k(y)=(H_{1k}(y),\dots,H_{nk}(y))^T$, $W_k(y)=(W_{1k}(y),\dots,W_{nk}(y))^T$ and $\tau(t)=(\tau_{ij}(t))$. For $i,j=1,\dots,n$, $k=1,2,\dots$, $c_i>0$, a_{ij},b_{ij},I_i,J_{ik} are constants, $f(y),g(y),H_k(y),W_k(y)\in C[R^n,R^n]$ and $\tau_{ij}(t)\in C[R,[0,\tau]]$, where $\tau>0$.

Definition 2.1. For any given $t_0\in R$, $\phi\in PC$, a function $y(t)\in PC[[t_0-\tau,+\infty),R^n]$ is called a solution of (2) through (t_0,ϕ), if $y(t)$ satisfies the initial condition in form

$$y(t_0+s)=\phi(s),\quad s\in[-\tau,0],\tag{3}$$

and satisfies Eq. (2) for $t\geqslant t_0$, denoted by $y(t,t_0,\phi)$. Especially, a point y^* in R^n is called an equilibrium point of (2), if $y(t)=y^*$ is a solution of (2).

Throughout this paper, we assume that for any $\phi\in PC$, there exists at least one solution with the initial values ϕ of Eq. (1). Let y^* be an equilibrium point of Eq. (2) and $y(t)$ be any solution of (2). Let $x(t)=y(t)-y^*$, substituting them into Eq. (1), we can get

$$\begin{cases} x'(t)=-Cx(t)+Af(x(t))+Bg(x(t-\tau(t))), & t\neq t_k,\\ x(t)=h_k(x(t^-))+w_k(x((t-\tau(t))^-)), & t=t_k,\end{cases}\tag{4}$$

where $f(x(t))=F(x(t)+y^*)-F(y^*)$, $g(x(t-\tau(t)))=G(x(t-\tau(t))+y^*)-G(y^*)$, $h_k(x(t))=H_k(x(t)+y^*)-H_k(y^*)$, $w_k(x(t-\tau(t)))=W_k(x(t-\tau(t))+y^*)-W_k(y^*)$.

It is clear that the stability of the zero solution of Eq. (4) is equivalent to the stability of the equilibrium point y^* of Eq. (1). Therefore, we may mainly discuss the stability of the zero of system (4).

Definition 2.2. The zero solution of Eq. (4) is said to be globally exponentially stable if for any solution $x(t,t_0,\phi)$ with the initial condition $\phi\in PC$, there exist constants $\alpha>0$ and $\kappa\geqslant 1$ such that

$$\|x(t,t_0,\phi)\|\leqslant\kappa\|\phi\|e^{-\alpha(t-t_0)},\quad t\geqslant t_0.\tag{5}$$

Definition 2.3 [32]. Let the matrix $D=(d_{ij})_{n\times n}$ have nonpositive off-diagonal elements (i.e., $d_{ij}\leqslant 0$, $i\neq j$), then each of the following conditions is equivalent to the statement "D is a nonsingular M-matrix."

(i) All the leading principle minors of D are positive.
(ii) $D=C-M$ and $\rho(C^{-1}M)<1$, where $M\geqslant 0$, $C=\operatorname{diag}\{c_1,\dots,c_n\}$ and $\rho(\cdot)$ is the spectral radius of the matrix $(\cdot)$.
(iii) The diagonal elements of D are all positive and there exists a positive vector d such that $Dd>0$ or $D^Td>0$.

Especially, the matrix D is a nonsingular M-matrix if it is row or column strictly dominant diagonal, that is, $De_n>0$ or $D^Te_n>0$.

For a nonsingular M-matrix D, we denote

$$\Omega_M(D)\triangleq\{z\in R^n\mid Dz>0,\ z>0\},$$

which is a nonempty set by (iii) of Definition 2.3, and satisfying that $k_1z_1+k_2z_2 \in \Omega_M(D)$ for any scalars $k_1, k_2 > 0$ and vectors $z_1, z_2 \in \Omega_M(D)$. So $\Omega_M(D)$ is a cone without vertex in R^n. We call it an "M-cone."

For a nonnegative matrix $A \in R^{n\times n}$, let $\rho(A)$ be the spectral radius of A. Then $\rho(A)$ is an eigenvalue of A and its eigenspace is denoted by

$$\Omega_\rho(A) \triangleq \{z \in R^n \mid Az = \rho(A)z\},$$

which includes all positive eigenvectors of A provided that the nonnegative matrix A has at least one positive eigenvector (see Refs. [32,33]).

3. Delay impulsive differential inequality

It is well known that differential inequalities are the main tools for studying the continuous differential systems. The following Halanay inequality has widely been applied to the stability analysis (see Refs. [25–30]).

Halanay inequality [21,22]. Let γ and p be constants with $0 < p < \gamma$. Let v be a continuous nonnegative scalar function on $[t_0-\tau, \beta]$ satisfying the inequality

$$v'(t) \leqslant -\gamma v(t) + p[v(t)]_\tau^+ \quad \text{for } t_0 \leqslant t < \beta. \tag{6}$$

Then

$$v(t) \leqslant [v(t_0)]_\tau^+ e^{\lambda(t-t_0)} \quad \text{for } t_0 \leqslant t < \beta, \tag{7}$$

where λ is a unique positive root of the equation $\lambda - p + qe^{\lambda\tau} = 0$.

Various generalized Halanay inequalities have presented by Baker and Tang [23], Tokumaru et al. [24], Wang and Xu [26], Xu [27] and Tian [30]. But, all of these inequalities cannot be applied to the impulsive differential systems since they require that $v(t)$ is continuous and the exponential estimate (7) includes the initial condition. In the impulsive differential systems, we need to estimate every continuous part on $[t_k, t_{k+1})$ with its initial function on $[t_k, t_k-\tau]$ for $k = 1, 2, \ldots$. This leads to that it is very difficult to get the estimate (5). To overcome these difficulties we use the properties of M-cone and eigenspace of the spectral radius of nonnegative matrices to introduce the following delay differential inequality with impulsive initial condition.

Theorem 3.1. *Let $P = (p_{ij})_{n\times n}$ and $p_{ij} \geqslant 0$ for $i \neq j$, $Q = (q_{ij})_{n\times n} \geqslant 0$ and $D = -(P+Q)$ be a nonsingular M-matrix. For $b \in (t_0, +\infty)$, let $u(t) = (u_1(t), \ldots, u_n(t))^T \in C[[t_0, b), R^n]$ be a solution of the following delay differential inequality with the initial condition $u(s) \in PC$, $t_0 - \tau \leqslant s \leqslant t_0$:*

$$D^+u(t) \leqslant Pu(t) + Q[u(t)]_\tau, \quad t \geqslant t_0. \tag{8}$$

Then

$$u(t) \leqslant ze^{-\lambda(t-t_0)}, \quad t \geqslant t_0, \tag{9}$$

provided that the initial conditions satisfies

$$u(s) \leqslant z e^{-\lambda(s-t_0)}, \quad t_0 - \tau \leqslant s \leqslant t_0, \tag{10}$$

where $z = (z_1, \dots, z_2)^T \in \Omega_M(D)$ *and the positive number* λ *is determined by the following inequality*:

$$[\lambda E + P + Q e^{\lambda\tau}] z < 0. \tag{11}$$

Proof. Since D is a nonsingular M-matrix, there exists a vector $z \in \Omega_M(D)$ such that

$$Dz > 0 \quad \text{or} \quad [P + Q]z < 0.$$

By using continuity, we obtain that Eq. (11) has at least one positive solution λ.

We at first shall prove that for any positive constant ϵ,

$$u_i(t) \leqslant (1+\epsilon) z_i e^{-\lambda(t-t_0)} \triangleq y_i(t), \quad t \geqslant t_0,\ i = 1, \dots, n. \tag{12}$$

If this is not true, from (10), then there must be a positive constant $t^* > t_0$ and some integer m such that

$$u_m(t^*) = y_m(t^*), \qquad D^+ u_m(t^*) \geqslant y'_m(t^*), \tag{13}$$

$$u_i(t) \leqslant y_i(t), \quad t \in [t_0 - \tau, t^*],\ i = 1, \dots, n. \tag{14}$$

By using (8), (12), (13), (14) and $p_{ij} \geqslant 0$ $(i \neq j)$, $Q \geqslant 0$, we obtain that

$$\begin{aligned} D^+ u_m(t^*) &\leqslant \sum_{j=1}^{n} \big[p_{mj} u_j(t^*) + q_{mj} \big[u_j(t^*)\big]_\tau \big] \\ &\leqslant \sum_{j=1}^{n} \big[p_{mj}(1+\epsilon) z_j e^{-\lambda(t^*-t_0)} + q_{mj}(1+\epsilon) z_j e^{-\lambda(t^*-\tau-t_0)} \big] \\ &= \sum_{j=1}^{n} [p_{mj} + q_{mj} e^{\lambda\tau}] z_j (1+\epsilon) e^{-\lambda(t^*-t_0)}. \end{aligned} \tag{15}$$

From (11), we have $\sum_{j=1}^{n} [p_{mj} + q_{mj} e^{\lambda\tau}] z_j < -\lambda z_m$. Then (15) becomes that

$$D^+ u_m(t^*) < -\lambda z_m (1+\epsilon) e^{-\lambda(t^*-t_0)} = y'_m(t^*),$$

which contradicts the inequality in (13). Thus (12) holds for all $t \geqslant t_0$. Therefore, letting $\epsilon \to 0$, we have

$$u_i(t) \leqslant z_i e^{-\lambda(t-t_0)}, \quad i = 1, \dots, n,\ t \geqslant t_0,$$

and the proof is complete. □

Remark 3.1. In the next section, we can get the estimate (5) by using Theorem 3.1 since the estimate (9) only include the information of M-cone determined by the coefficient matrices P and Q provided the initial function satisfies the same exponential estimate.

4. Global exponential stability

In this section, we will give several sufficient conditions on the global exponential stability of equilibrium point for the neural network (1).

Theorem 4.1. *Assume that*

(A1) *For any $x \in R^n$ there exist nonnegative diagonal matrices U, V such that*

$$\big[f(x)\big]^+ \leqslant U[x]^+, \qquad \big[g(x)\big]^+ \leqslant V[x]^+. \tag{16}$$

(A2) *For any $x \in R^n$ there exist nonnegative matrices R_k, S_k such that*

$$\big[h_k(x)\big]^+ \leqslant R_k[x]^+, \quad \big[w_k(x)\big]^+ \leqslant S_k[x]^+, \quad k = 1, 2, \ldots. \tag{17}$$

(A3) *Let $P = -C + [A]^+U$, $Q = [B]^+V$ and $D = -(P + Q)$ be a nonsingular M-matrix.*
(A4) *The set $\Omega = \bigcap_{k=1}^{\infty}[\Omega_\rho(R_k) \cap \Omega_\rho(S_k)] \cap \Omega_M(D)$ is nonempty (i.e., $\Omega \neq \emptyset$).*
(A5) *There exists a constant γ such that*

$$\frac{\ln \gamma_k}{t_k - t_{k-1}} \leqslant \gamma < \lambda, \quad k = 1, 2, \ldots, \tag{18}$$

where the scalar $\lambda > 0$ is determined by the inequality

$$[\lambda E + P + Qe^{\lambda\tau}]z < 0 \quad \text{for a given } z \in \Omega, \tag{19}$$

and

$$\gamma_k \geqslant \max\big\{1, \rho(R_k) + \rho(S_k)e^{\lambda\tau}\big\}. \tag{20}$$

Then the zero solution of (4) *is globally exponentially stable and the exponential convergent rate equals $\lambda - \gamma$.*

Proof. At first calculating the upper right derivative $D^+[x(t)]^+$ along the solutions of Eq. (4), from (16) we have

$$\begin{aligned} D^+\big[x(t)\big]^+ &= \mathrm{Sgn}(x)x' \\ &\leqslant -C\big[x(t)\big]^+ + \big[Af\big(x(t)\big)\big]^+ + \big[Bg\big(x\big(t - \tau(t)\big)\big)\big]^+ \\ &\leqslant -C\big[x(t)\big]^+ + [A]^+U\big[x(t)\big]^+ + [B]^+V\big[x(t)\big]_\tau^+ \\ &\leqslant P\big[x(t)\big]^+ + Q\big[x(t)\big]_\tau^+, \quad t_{k-1} \leqslant t < t_k,\ k = 1, 2, \ldots. \end{aligned} \tag{21}$$

Since D is a nonsingular M-matrix and the set Ω is nonempty, there exists a vector $z \in \Omega \subset \Omega_M(D)$ such that

$$Dz > 0 \quad \text{or} \quad [P + Q]z < 0.$$

By using continuity, we obtain that Eq. (19) has at least one positive solution λ.

For the initial conditions: $x(t_0 + s) = \phi(s)$, $s \in [-\tau, 0]$, where $\phi \in PC$ and $t_0 \in R$ (no loss of generality, we assume $t_0 \leqslant t_1$), we can get

$$\big[x(t)\big]^+ \leqslant d\|\phi\|e^{-\lambda(t-t_0)}, \quad t_0 - \tau \leqslant t \leqslant t_0, \tag{22}$$

where

$$d=\frac{1}{\min_{1\leqslant i\leqslant n}\{z_i\}}z\geqslant e_n.$$

From the property of M-cone and $z\in\Omega\subseteq\Omega_M(D)$, we have $d\|\phi\|\in\Omega_M(D)$. Then, all conditions of Theorem 3.1 are satisfied by (21), (22) and condition (A3), so

$$\big[x(t)\big]^+\leqslant d\|\phi\|e^{-\lambda(t-t_0)},\quad t_0\leqslant t<t_1. \tag{23}$$

Suppose that for all $m=1,\dots,k$ the inequalities

$$\big[x(t)\big]^+\leqslant \gamma_0\dots\gamma_{m-1}d\|\phi\|e^{-\lambda(t-t_0)},\quad t_{m-1}\leqslant t<t_m, \tag{24}$$

hold, where $\gamma_0=1$. Then, from (17) and (24), the discrete part of (4) satisfies that

$$\begin{aligned}\big[x(t_k)\big]^+&\leqslant\big[h_k\big(x\big(t_k^-\big)\big)\big]^+ +\big[w_k\big(x\big((t-t_k)^-\big)\big)\big]^+\\&\leqslant R_k\big[x\big(t_k^-\big)\big]^+ +S_k\big[x\big(\big(t_k-\tau(t_k)\big)^-\big)\big]^+\\&\leqslant R_k\gamma_0\dots\gamma_{k-1}d\|\phi\|e^{-\lambda(t_k-t_0)}+S_k\gamma_0\dots\gamma_{k-1}d\|\phi\|e^{-\lambda(t_k-\tau-t_0)}\\&\leqslant[R_kd+S_kde^{\lambda\tau}]\gamma_0\dots\gamma_{k-1}\|\phi\|e^{-\lambda(t_k-t_0)}.\end{aligned} \tag{25}$$

Since $d\in\Omega\subseteq\Omega_\rho(R_k)\cap\Omega_\rho(S_k)$, we have $R_kd=\rho(R_k)d$ and $S_kd=\rho(S_k)d$. Therefore, from (20) and (25), we obtain

$$\begin{aligned}\big[x(t_k)\big]^+&\leqslant\big[\rho(R_k)+\rho(S_k)e^{\lambda\tau}\big]d\gamma_0\dots\gamma_{k-1}\|\phi\|e^{-\lambda(t_k-t_0)}\\&\leqslant\gamma_0\dots\gamma_{k-1}\gamma_kd\|\phi\|e^{-\lambda(t_k-t_0)}.\end{aligned} \tag{26}$$

This, together with (24), lead to

$$\big[x(t)\big]^+\leqslant\gamma_0\dots\gamma_{k-1}\gamma_kd\|\phi\|e^{-\lambda(t-t_0)}\quad\text{for all } t\in[t_k-\tau,t_k]. \tag{27}$$

By the property of M-cone again, the vector $\gamma_0\dots\gamma_{k-1}\gamma_kd\|\phi\|\in\Omega_M(D)$. It follows from (27) and Theorem 3.1 that

$$\big[x(t)\big]^+\leqslant\gamma_0\dots\gamma_{k-1}\gamma_kd\|\phi\|e^{-\lambda(t-t_0)},\quad t_k\leqslant t<t_{k+1}. \tag{28}$$

By the mathematical induction, we can conclude that

$$\big[x(t)\big]^+\leqslant\gamma_0\dots\gamma_{k-1}d\|\phi\|e^{-\lambda(t-t_0)},\quad t_{k-1}\leqslant t<t_k,\ k=1,2,\dots. \tag{29}$$

Noticing that $\gamma_k\leqslant e^{\gamma(t_k-t_{k-1})}$ by (18), we can use (29) to conclude that

$$\begin{aligned}\big[x(t)\big]^+&\leqslant e^{\gamma(t_1-t_0)}\dots e^{\gamma(t_{k-1}-t_{k-2})}d\|\phi\|e^{-\lambda(t-t_0)}\leqslant d\|\phi\|e^{\gamma(t-t_0)}e^{-\lambda(t-t_0)}\\&=d\|\phi\|e^{-(\lambda-\gamma)(t-t_0)},\quad\forall t\in[t_0,t_k),\ k=1,2,\dots.\end{aligned}$$

This implies that the conclusion of the theorem hold. □

Replacing M-matrix in Theorem 4.1 with strictly dominant diagonal matrix, we can obtain the following Theorem 4.2 which does not require condition (A4).

Theorem 4.2. *Assume that the matrix D in* (A3) *is a row strictly dominant diagonal matrix and the conditions* (A1), (A2) *and* (A5) *with*

$$z = e_n, \qquad \gamma_k \geqslant \max\{1, \|R_k + V S_k e^{\lambda\tau}\|\} \quad \textit{and} \quad (\lambda E + P + Q e^{\lambda\tau}) e_n < 0 \tag{30}$$

hold. Then the zero solution of (4) *is globally exponentially stable and exponentially convergent rate equals $\lambda - \gamma$.*

Proof. Since D is row strictly dominant diagonal, we have

$$[P + Q] e_n < 0.$$

By using continuity, we obtain that the strict inequality in (30) has at least one positive solution λ. The remainder of the proof of Theorem 4.2 is essentially the same as the proof of Theorem 4.1 except that one chooses $z = e_n$ and notes that the inequality

$$[x(t)]^+ \leqslant \gamma_0 \ldots \gamma_{m-1} e_n \|\phi\| e^{-\lambda(t-t_0)}, \quad t_{m-1} \leqslant t < t_m,\ m = 1, \ldots, k, \tag{31}$$

can imply

$$\begin{aligned}
[x(t_k)]^+ &\leqslant [h_k(x(t_k))]^+ + [w_k(x((t_k - \tau(t_k))^-))]^+ \\
&\leqslant R_k [x(t_k^-)]^+ + S_k [x((t_k - \tau(t_k))^-)]^+ \\
&\leqslant R_k \gamma_0 \ldots \gamma_{k-1} e_n \|\phi\| e^{-\lambda(t_k - t_0)} + S_k \gamma_0 \ldots \gamma_{k-1} e_n \|\phi\| e^{-\lambda(t_k - \tau - t_0)} \\
&\leqslant [R_k + S_k e^{\lambda\tau}] e_n \gamma_0 \ldots \gamma_{k-1} \|\phi\| e^{-\lambda(t_k - t_0)} \\
&\leqslant \|R_k + S_k e^{\lambda\tau}\| e_n \gamma_0 \ldots \gamma_{k-1} \|\phi\| e^{-\lambda(t_k - t_0)} \\
&\leqslant \gamma_0 \ldots \gamma_{k-1} \gamma_k e_n \|\phi\| e^{-\lambda(t_k - t_0)},
\end{aligned} \tag{32}$$

by using (17) and (30). □

5. Corollaries and remarks

In Theorem 4.1, we may properly choose the matrices R_k and S_k in the inequality (17) to guarantee $\Omega \neq \emptyset$. Especially, when $R_k = \alpha_k E$ and $S_k = \beta_k E$ (α_k, β_k are nonnegative constants), Ω is certainly nonempty. Therefore, by employing Theorem 4.1, we easily obtain the following corollary.

Corollary 5.1. *Assume that conditions* (A1), (A3), (A5) *hold. For any $x \in R^n$ there exist nonnegative constants α_k, β_k, such that*

$$[h_k(x)]^+ \leqslant \alpha_k [x]^+, \quad [w_k(x)]^+ \leqslant \beta_k [x]^+, \quad k = 1, 2, \ldots. \tag{33}$$

And let $\gamma_k \geqslant \max\{1, \alpha_k + \beta_k e^{\lambda\tau}\}$, where the scalar $\lambda > 0$ is determined by the inequality (19) *with $z \in \Omega_M(D)$. Then the zero solution of* (4) *is globally exponentially stable and the exponential convergent rate equals $\lambda - \gamma$.*

Proof. Noting that (33) is a special case of (17) with $R_k = \alpha_k E$ and $S_k = \beta_k E$, we get (A2) and

$$\rho(R_k) = \alpha_k, \quad \rho(S_k) = \beta_k, \quad \Omega_\rho(R_k) = \Omega_\rho(S_k) = R^n,$$
$$\Omega = \bigcap_{k=1}^{\infty} \big[\Omega_\rho(R_k) \cap \Omega_\rho(S_k)\big] \cap \Omega_M(D) = \Omega_M(D).$$

Since the M-cone $\Omega_M(D)$ is nonempty, (A4) obviously holds. By using Theorem 4.1 we can deduce the conclusion. □

If $H_k(y) = y$, $W_k(y) = 0$, $J_k = 0$, then the model (2) becomes delay neural networks without impulses in vector form

$$y' = -Cy(t) + AF\big(y(t)\big) + BG\big(y\big(t - \tau(t)\big)\big) + I. \tag{34}$$

By using Corollary 5.1, we can obtain the stability of the equilibrium point of Eq. (34).

Corollary 5.2. *Assume that* (A3) *holds and its parameters are determined by the system* (34) *and the following condition*:

(A1′) *For any $x, y \in R^n$ there exist nonnegative diagonal matrices U, V such that*

$$\big[F(x) - F(y)\big]^+ \leqslant U[x-y]^+, \quad \big[G(x) - G(y)\big]^+ \leqslant V[x-y]^+. \tag{35}$$

Then Eq. (34) *has exactly one equilibrium point, which is globally exponentially stable.*

Remark 5.1. (A1′) is a vector form of the globally Lipschitz conditions used by many researchers [3–6,8,11,12]. It is obvious that (A1) holds if (A1′) holds.

Proof. We first shall prove the existence of the equilibrium point of (34), which is a solution of the algebra equation

$$-Cu + AF(u) + BG(u) + I = 0. \tag{36}$$

Define operator $T(u) = C^{-1}(AF(u) + BG(u) + I)$, $u \in R^n$. From (35), we have

$$\begin{aligned}
\big[T(u)\big]^+ &\leqslant C^{-1}\big\{[A]^+\big[F(u)\big]^+ + [B]^+\big[G(u)\big]^+ + [I]^+\big\} \\
&\leqslant C^{-1}\big\{[A]^+\big(U[u]^+ + \big[F(0)\big]^+\big) + [B]^+\big(V[u]^+ + \big[G(0)\big]^+\big) + [I]^+\big\} \\
&= C^{-1}\big\{[A]^+U + [B]^+V\big\}[u]^+ + C^{-1}J,
\end{aligned} \tag{37}$$

where the constant vector $J = [A]^+[F(0)]^+ + [B]^+[G(0)]^+ + [I]^+$. Since D is a nonsingular M-matrix by condition (A3), from (iii) of Definition 2.3, there exists a positive vector d such that

$$J \leqslant Dd = \big(C - [A]^+U - [B]^+V\big)d,$$

yielding

$$C^{-1}J \leqslant \big(E - C^{-1}\big([A]^+U + [B]^+V\big)\big)d \quad \text{or}$$
$$C^{-1}\big\{[A]^+U + [B]^+V\big\}d + C^{-1}J \leqslant d. \tag{38}$$

Let $\mathcal{B} = \{u \in R^n \mid [u]^+ \leqslant d\}$. From (37), (38), for any $u \in \mathcal{B}$ we have $[T(u)]^+ \leqslant d$. So the continuous operator T maps the close set $\mathcal{B}$ into itself. By Brouwer's fixed point theorem, T has a fixed point $u^* \in \mathcal{B}$, which is the equilibrium point of (34).

We next show the equilibrium u^* is unique. Let $\hat{u}$ be any equilibrium of (36), then

$$u^* = C^{-1}\big[AF(u^*) + BG(u^*) + I\big], \qquad \hat{u} = C^{-1}\big[AF(\hat{u}) + BG(\hat{u}) + I\big]. \tag{39}$$

So, we derive

$$\begin{aligned}[u^* - \hat{u}]^+ &\leqslant [C^{-1}A]^+\big[F(u^*) - F(\hat{u})\big]^+ + [C^{-1}B]^+\big[G(u^*) - G(\hat{u})\big]^+ \\ &\leqslant C^{-1}\big([A]^+U + [B]^+V\big)[u^* - \hat{u}]^+. \end{aligned} \tag{40}$$

If $[u^* - \hat{u}]^+ \neq 0$, then, by Theorem 8.3.2 of [33] and $[u^* - \hat{u}]^+ \geqslant 0$, we have $\rho(C^{-1}([A]^+U + [B]^+V)) \geqslant 1$. However, condition (A3) and (ii) in Definition 2.3 imply that $\rho(C^{-1}([A]^+U + [B]^+V)) < 1$. This is a contradiction. Hence, $\hat{u} = u^*$ and system (34) has a unique equilibrium point u^*. The globally exponentially stability of the equilibrium can be directly implied by Corollary 5.1. The proof is complete. □

Remark 5.2. A condition of equivalence has been given in [4] for the system (34) with $F \equiv 0$. And Corollary 5.2 is an generalization or improvement of some recent results (e.g., [3, Theorem 2.4], [4, Theorems 2 and 5], [5, Theorem 1], [6, Theorem 2.3], [8, Theorems 1 and 2], [11, Theorems 1, 3 and 4], [12, Theorem 2], etc.).

Remark 5.3. In general, ones always assume that there is an equilibrium point for the impulsive systems to study their stability. However, Corollary 5.2 shows that there is a unique equilibrium point u^* of the continuous part of the system (1) under the conditions (A1$'$) and (A3). In many cases, u^* may not be a solution of the discrete part of the system (1) without the external impulsive input. That is, the entire system (1) may have no equilibrium point. In order to guarantee that the entire system (1) has an equilibrium point, we introduce the external impulsive input J_k so that u^* is also an equilibrium point of the discrete part of the system (1).

6. Illustrative example

The following illustrative example will demonstrate the effectiveness of our results.

Example 6.1. Consider the impulsive delay neural networks

$$\begin{cases} y_1'(t) = -6y_1(t) + 2y_1(t) - y_2(t) + |y_1(t - \tau_{11}(t))| + 4, \\ y_2'(t) = -5y_2(t) + 2.5y_1(t) + 2y_2(t) - 0.5|y_1(t - \tau_{21}(t))| + |y_2(t - \tau_{22}(t))|, \\ t \neq t_k, \end{cases} \tag{41}$$

with

$$\begin{cases} y_1(t_k) = H_{1k}(y_1(t_k^-), y_2(t_k^-)) \\ \qquad\qquad + W_{1k}(y_1((t_k - \tau_{11}(t_k))^-), y_2((t_k - \tau_{12}(t_k))^-)) + J_{1k}, \\ y_2(t_k) = H_{2k}(y_1(t_k^-), y_2(t_k^-)) \\ \qquad\qquad + W_{21}(y_1((t_k - \tau_{21}(t_k))^-), y_2((t_k - \tau_{22}(t_k))^-)) + J_{2k}, \end{cases} \tag{42}$$

where $\tau_{ij}(t) = |\sin((i+j)t)| \leqslant 1 \triangleq \tau$ for $i, j = 1, 2$, and $t_1 = 0.3$, $t_k = t_{k-1} + 0.3k$ for $k = 2, 3, \ldots$.

(i) If $H_{ik}(y_1, y_2) = y_i$, $W_{ik}(y_1, y_2) = 0$, $J_{ik} = 0$ for $i = 1, 2$ and $k = 1, 2, \ldots$, then the system (41) becomes delay neural networks without impulses. The parameters of conditions (A1′) and (A3) are as follows:

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P = \begin{pmatrix} -4 & 1 \\ 2.5 & -3 \end{pmatrix},$$
$$Q = \begin{pmatrix} 1 & 0 \\ 0.5 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 3 & -1 \\ -3 & 2 \end{pmatrix},$$

where $P = -C + [A]^+U$, $Q = [B]^+V$, $D = -(P + Q)$. We easily observe D is a nonsingular M-matrix. By Corollary 5.2 the system has exactly one globally exponentially stable equilibrium point, which is actually $(1, 1)^T$.

(ii) If

$$H_{1k}(y_1, y_2) = 0.2e^{0.05k}y_1 - 0.1e^{0.05k}y_2,$$
$$H_{2k}(y_1, y_2) = -0.4e^{0.05k}y_1 + 0.2e^{0.05k}y_2,$$
$$W_{11}(y_1, y_2) = 0.4e^{0.05k}y_1, \qquad W_{22}(y_1, y_2) = -0.4e^{0.05k}y_2,$$
$$J_1 = 1 - 0.5e^{0.05k}, \qquad J_2 = 1 + 0.6e^{0.05k},$$

we can verify that the point $(1, 1)^T$ is also an equilibrium point of the delay impulsive system (41) with (42), and the parameters of conditions (A2) and (A4) are as follows:

$$R_k = 0.1e^{0.05k}\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}, \qquad \rho(R_k) = 0.4e^{0.05k},$$
$$S_k = 0.2e^{0.05k}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \rho(S_k) = 0.4e^{0.05k},$$
$$\Omega_\rho(R_k) = \{(z_1, z_2)^T \mid z_2 = 2z_1\}, \qquad \Omega_\rho(S_k) = R^2,$$
$$\Omega_M(D) = \{(z_1, z_2)^T > 0 \mid 2z_1/3 < z_2 < 4z_1\}.$$

So $\Omega = \{(z_1, z_2)^T \mid z_2 = 2z_1\}$ is not nonempty. Let $d = (1, 2)^T \in \Omega$ and $\lambda = 0.2$ which satisfies the inequality $(\lambda E + P + Qe^{\lambda\tau})d < 0$. We can obtain that for $k = 1, 2, \ldots$,

$$\gamma_k = e^{0.05k} \geqslant \max\{1, 0.4e^{0.05k} + 0.4e^{0.05k}e^{0.2}\},$$
$$\frac{\ln \gamma_k}{t_k - t_{k-1}} \leqslant \frac{\ln e^{0.05k}}{0.3k} \leqslant 0.1667 = \gamma < \lambda.$$

Clearly, all conditions of Theorem 4.1 are satisfied, so the equilibrium point is globally exponentially stable and the exponentially convergent rate is approximately equal to 0.04.

Figure 1 shows the simulation results of the global exponential stability of the system (41), (42) with the initial conditions: $y_1(s) = \tanh(s)$, $y_2(s) = e^{(s+0.5)}$, $s \in [-1, 0]$.

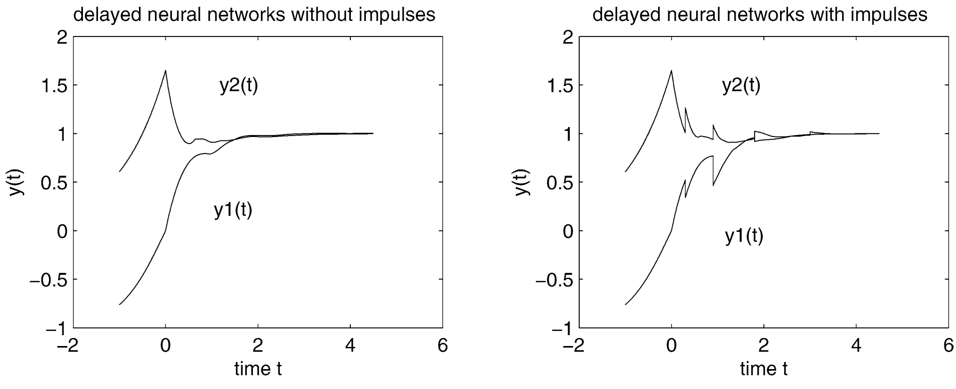

Fig. 1. Stability for delay neural networks.

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